Advanced Engineering Mathematics
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Lecture Notes
Second Week

1 Functions of two or more than one variable

In a 2-dimensional space a point is represented by an ordered pair (x, y) and the set of all such points constitutes a 2-dimensional Euclidean space $\mathbb{R}^2$. For $\mathbb{R}^n$, a point will be of form $(x_1, x_2, ..., x_n)$. If in 2D space the variables x and y are independent such that a variable $z = f(x, y)$ is a function of x and y. Here z is dependent variable. The subset D of $\mathbb{R}^2$ is the set of points for which the function $f(x, y)$ is defined, then D is called domain of f.

Example 1.1. $z = \frac{xy}{x^2-y^2}$. Here z is not defined for $x = y$.
$D = \{(x, y) : x \neq y\} = \{(x, y) \in \mathbb{R}^2 : x \neq y\} = \{x \in \mathbb{R}, y \in \mathbb{R} : x \neq y\}$

Example 1.2. $z = \sqrt{1 - x - y}$. Here $D = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$

Limit of $f(x, y)$. Let D be a subset of $\mathbb{R}^2$ which is domain of definition of a function f. Let (a, b) be a point in D. Let $(x, y) \in D$ and $(x, y) \to (a, b)$ in any manner. f is said to tend to a limit A if $\forall \varepsilon > 0\ \exists\ \delta > 0$ such that $|f(x, y) - A| < \varepsilon$, whenever $(x, y) \in$ a nbd of the point (a, b). We may choose the nbd as

$$N_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : 0 < |x - a| < \delta, 0 < |y - b| < \delta\} \quad \text{or,}$$
$$N_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \delta^2\}$$

Then the limit, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = A$ is called double limit or simultaneous limit.

Iterated or Repeated Limit. Let N be an nbd of (a, b) and $f(x, y)$ is defined on N. For a fixed value of y, $\lim\limits_{x\to a} f(x, y)$, if exists, will involve y. Therefore, $\lim\limits_{x\to a} f(x, y)$, will be different for different values of y. That is $\lim\limits_{x\to a} f(x, y) = \phi(y)$. Again if $\lim\limits_{y\to b} \phi(y)$, exists and equal to A, then we write $\lim\limits_{y\to b} \lim\limits_{x\to a} f(x, y) = A$.

Example 1.3. Find the limit $\lim\limits_{x\to a} f(x, y)$, where $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$.

Sol. Let us take $x = r\cos\theta, y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and as $(x, y) \to (0, 0)$, $r \to 0$
$f(x, y) = r^2 \sin\theta\cos\theta\cos 2\theta$,

$$|xy\frac{x^2 - y^2}{x^2 + y^2}| = |\frac{r^2}{4}\sin 4\theta| \leq \frac{r^2}{4} < \varepsilon, \quad \text{if} \quad r^2 < 4\varepsilon \quad \text{or} \quad 0 < x^2 + y^2 < \delta^2.$$

So, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ (using definition).

Example 1.4. Let, $f(x, y) = \begin{cases} x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$
Show that $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y)$ exists but neither of $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ nor $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x, y)$ exists.

Sol. We shall show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist. Since $(x, y) \to (0, 0)$, let, $y = mx$ be the path.

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$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1=m^2}{1+m^2} \ (x \neq 0)$$

So, the limit is path dependant, i.e., the limit is different for different values of m.
Again, $\lim_{y\to 0} \sin\frac{1}{y}$ does not exist, then $\lim_{x\to 0}\lim_{y\to 0} x\sin\frac{1}{y}$ does not exist therefore, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Now, $\lim_{x\to 0} x\sin\frac{1}{y} = 0$ and $\lim_{x\to a} f(x,y) = -1$, then $\lim_{y\to 0}(0-1) = -1$
,i.e., $\lim_{y\to 0}\lim_{x\to 0}(x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2}) = -1$

Partial Derivatives for functions of two or more variables. $D \subset \mathbb{R}$ is usual domain of f and let $P(a,b) \in D$ be arbitrary. Let N be an nbd and $Q(a+h, b+k) \in N$. Then if α be an angle that the line joining $\vec{PQ}$ makes with positive $x-$axis, then the direction cosines of $\vec{PQ}$ are $l = \cos\alpha, m = \sin\alpha$. If $\rho = \sqrt{h^2+k^2}$. Then $h = \rho\cos\alpha, k = \rho\sin\alpha$ and $\rho \to 0$ as $Q \to P$. Now if,

$$\lim_{\rho\to 0} \frac{f(a+p\cos\alpha, b+p\sin\alpha) - f(a,b)}{\rho} \quad \text{exists.}$$

Then, it is called derivative of $f(x,y)$ at (a,b) in direction of α and it is denoted by $D_\alpha f(a,b)$. If $\alpha = 0$, then derivative is denoted by $\frac{\partial f}{\partial x}$ and if $\alpha = \frac{\pi}{2}$, then the derivative is denoted by $\frac{\partial f}{\partial y}$. These derivatives are called partial derivatives.
Continuity of $f(x,y)$. Let $N \subset D \subset \mathbb{R}^2$, where D is the domain of and N is the nbd of a point $(a,b) \in D$. Then f is said to be continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Equivalently, $\forall\varepsilon > 0\ \exists\ \delta > 0$ such that $|f(x,y) - f(a,b)| < \varepsilon$ whenever $|x-a| < \delta$ and $|y-b| < \delta$ or $(x-a)^2 + (y-b)^2 < \delta^2$
or,
$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$ where $(a+h, b+k) \in D$.

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Example 1.5. Verify the continuity of $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ at $(0,0)$.

Sol. Here,

$$|f(x,y) - f(0,0)| = |xy\frac{x^2-y^2}{x^2+y^2} - 0| = |xy\frac{x^2-y^2}{x^2+y^2}| = |x||y||\frac{x^2-y^2}{x^2+y^2}| \leq |x||y| \leq \frac{x^2+y^2}{2} < \varepsilon,$$

if $x^2+y^2 < \delta^2(=2\varepsilon) \Rightarrow f$ is continuous at $(0,0)$.

Example 1.6. Verify the continuity of $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$ at $(0,0)$.

Sol. Here,

$$|f(x,y) - f(0,0)| = |x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0| \leq |x||\sin\frac{1}{y} + |y|\sin\frac{1}{x} \leq |x| + |y| < \varepsilon$$

if $|x-0| < \delta$, and $|y-0| < \delta(=\frac{\varepsilon}{2}) \Rightarrow f$is continuous at $(0,0)$.

Example 1.7. Verify the continuity of $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$ at $(0,0)$.

Partial Derivatives (Computations). If $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$ exists, then it is denoted by $\frac{\partial f}{\partial x}$ or f_x similarly, if $\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$ exists, then it is denoted by $\frac{\partial f}{\partial y}$ or f_y.

Example 1.8. Let, $f(x,y)=x^2y+3xy$. Find f_x, f_y.

Sol. $f_x=2xy+3y$ and $f_y=x^2+3x$.

Let $z=f(x,y)$ then,

$$z_x=\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}, z_y=\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y},$$
$$z_{xx}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})=\frac{\partial^2 z}{\partial x^2}, z_{yy}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial y})=\frac{\partial^2 z}{\partial y^2},\ z_{yx}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=\frac{\partial^2 z}{\partial y\partial x}\cdots,$$
$$\frac{\partial}{\partial y}(\frac{\partial^{n-1}}{\partial y^{n-1}}z)=\frac{\partial^n z}{\partial y^n}, \frac{\partial}{\partial x}(\frac{\partial^{n-1}}{\partial x^{n-1}}z)=\frac{\partial^n z}{\partial x^n}.$$

Example 1.9. $z=f(x,y)=x^3+3xy^4-3x^4y^3$. Find z_x, z_y, z_{xx}, z_{yy}, z_{yx}, z_{xy}.

Sol.

$$\begin{aligned} z_x &= 3x^2+3y^4-12x^3y^3,\\ z_y &= 12xy^3-9x^4y^2,\\ z_{xx} &= 6x-36x^2y^3,\\ z_{yy} &= 36xy^2-24x^4y,\\ z_{yx} &= 12y^3-36x^3y^2,\\ z_{xy} &= 12y^3-36x^3y^2 \end{aligned}$$

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Taylor's Series: The Taylor's series of a function $f(x,y)$ about a point (x_0,y_0) provide an approximation of a function in the nbd of (x_0,y_0) and

$$z(x,y)=z(x_0,y_0)+\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}(x-x_0)+\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}(y-y_0)+\frac{1}{2}\Big(\frac{\partial^2 z}{\partial x^2}\Big|_{(x_0,y_0)}(x-x_0)^2 + \frac{\partial^2 z}{\partial y^2}\Big|_{(x_0,y_0)}(y-y_0)^2+2\frac{\partial^2 z}{\partial x\partial y}\Big|_{(x_0,y_0)}(x-x_0)(y-y_0)\Big)+...$$

Example 1.10. Let us take the function, $z(x,y)=x^3+3y-y^3-3x$. Find z_x, z_y, z_{xx}, z_{yy}, z_{yx}, z_{xy}.
Sol. $z_x=3x^2-3$, $z_y=3-3y^2$, $z_{xx}=6x$, $z_{yy}=-6y$, $z_{yx}=0$, $z_{xy}=0$.

2 Maxima and Minima

Stationary Points: Maxima and Minima:
If $f'(x)=0$ at $x=x_0$ and $f''(x)\Big|_{x=x_0}>0$ then f attains minimum value at $x=x_0$. Similarly, if $f'(x)=0$ at $x=x_0$ and $f''(x)\Big|_{x=x_0}<0$ then f attains maximum value at $x=x_0$.

If f is a function of two variables, $\frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial y}=0 \Rightarrow x=x_0, y=y_0$. Then we will calculate second order derivatives of it at point (x_0,y_0).

Let, $z_{xx}\Big|_{(x_0,y_0)} = A$, $z_{yy}\Big|_{(x_0,y_0)} = B$, $z_{xy}\Big|_{(x_0,y_0)} = C$. Compute the discriminant: $D = BC - A^2$

Case I. If $D > 0$ and $\frac{\partial^2 z}{\partial x^2}|_{(x_0,y_0)} > 0$ or $\frac{\partial^2 z}{\partial y^2}|_{(x_0,y_0)} > 0$, then z attains local minima at point (x_0, y_0).

Case II. If $D > 0$ and $\frac{\partial^2 z}{\partial x^2}|_{(x_0,y_0)} < 0$ or $\frac{\partial^2 z}{\partial y^2}|_{(x_0,y_0)} < 0$, then z attains local maxima at point (x_0, y_0).

Case III. If $D < 0$, then (x_0, y_0) is a saddle point.

Example 2.1. Find the maxima and minima of $f(x) = x^3 - 4x^2$.

Sol. Given

$$\begin{aligned} f(x) &= x^3 - 4x^2 \\ \Rightarrow f'(x) &= 3x^2 - 8x \end{aligned}$$

For extrema/ extremum,

$$f'(x) = 0 \Rightarrow 3x^2 - 8x = 0 \Rightarrow x = 0, x = \frac{8}{3}.$$

Again,

$$\begin{aligned} f''(x) &= 6x - 8 \\ f''(0) &= -8 < 0 \\ f''(\frac{8}{3}) &= 6 \times \frac{8}{3} - 8 = 16 - 8 = 8 > 0. \end{aligned}$$

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At, $x = 0, f''(x) < 0$ which implies that the function attains its maximum value at $x = 0$.
At, $x = \frac{8}{3}, f''(x) > 0$, which implies that the function attains its minimum value at $x = \frac{8}{3}$.
So, the maximum value of

$$\begin{aligned} f &= f(x)\Big|_{x=0} \\ &= 0 \end{aligned}$$

And the minimum value of

$$\begin{aligned} f &= f(x)\Big|_{x=\frac{8}{3}} \\ &= x^3 - 4x^2\Big|_{x=\frac{8}{3}} \\ &= \frac{8^3}{27} - \frac{4.8^2}{9} \\ &= \frac{8^2}{9} \times \Big[\frac{8}{3} - 4\Big] \\ &= \frac{8^2}{9} \times \Big(-\frac{4}{3}\Big) \\ &= -\frac{256}{27} \end{aligned}$$

Example 2.2. Find the maximum and minimum values of the function,

$$f(x, y) = x^3 - y^3 + 3y - 3x$$

Sol. Here,

$$f_x(x,y) = 3x^2 - 3$$
$$f_y(x,y) = -3y^2 + 3$$

Suppose the extremum values are attained at the point (x_0, y_0). Then for the extremum,

$$f_x(x_0,y_0) = 0 \Rightarrow 3x_0^2 - 3 = 0 \Rightarrow x_0^2 = 1 \Rightarrow x_0 = \pm 1$$
$$f_y(x_0,y_0) = 0 \Rightarrow -3y_0^2 + 3 = 0 \Rightarrow y_0^2 = 1 \Rightarrow y_0 = \pm 1$$

The extremum points are $P_1(1,1),\ P_2(1,-1),\ P_3(-1,1),\ P_4(-1,-1)$. For maximum and minimum values, we need the 2nd derivatives and the discriminant D,

$$f_{xx} = 6x, f_{yy} = -6y, f(x,y) = 0, f_{yx} = 0$$
$$D = AB - C^2 = 6x \times (-6y) - 0 = -36xy$$

Now,

$$D\Big|_{P_1(1,1)} = -36 \times 1 \times 1 = -36 < 0, A = 6, B = -6, C = 0 \rightarrow \text{Saddle Point}$$
$$D\Big|_{P_2(1,-1)} = 36 > 0, A = 6, B = 6, C = 0 \rightarrow \text{Minimum Value}$$
$$D\Big|_{P_3(-1,1)} = 36 > 0, A = -6, B = -6, C = 0 \rightarrow \text{Maximum Value}$$
$$D\Big|_{P_4(-1,-1)} = -36 < 0, A = -6, B = 6, C = 0 \rightarrow \text{Saddle Point}$$
$$f(x,y)\Big|_{(1,-1)} = 8, f(x,y)\Big|_{(-1,1)} = 4$$

Example 2.3. For the function

$$f(x,y) = x^3 - y^3 + 3y - 3x$$

Find out the Taylor's series expansion about the point $P(2,1)$.

Sol.

$$f(x,y) = x^3 - y^3 + 3y - 3x \Rightarrow f(x,y)\Big|_{P(2,1)} = 4$$
$$f_x(x,y) = 3x^2 - 3 \Rightarrow f_x(x,y)\Big|_{P(2,1)} = 9$$
$$f_y(x,y) = -3y^2 + 3 \Rightarrow f_y(x,y)\Big|_{P(2,1)} = 0$$
$$f_{xx}(x,y) = 6x \Rightarrow f_{xx}(x,y)\Big|_{P(2,1)} = 12$$
$$f_{yy}(x,y) = -6y \Rightarrow f_{yy}(x,y)\Big|_{P(2,1)} = -6$$
$$f_{xy}(x,y) = 0 \Rightarrow f_{xy}(x,y)\Big|_{P(2,1)} = 0$$

By Taylor's series,

$$f(x,y) = f((2,1)) + \frac{\partial f}{\partial x}\Big|_{(2,1)}(x-2) + \frac{\partial f}{\partial y}\Big|_{(2,1)}(y-1) + \frac{1}{2}\Big(\frac{\partial^2 z}{\partial x^2}\Big|_{(2,1)}(x-2)^2 + \frac{\partial^2 z}{\partial y^2}\Big|_{(2,1)}(y-1)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\Big|_{(2,1)}(x-2)(y-1)\Big) + \ldots$$
$$= 4 + 9 \times (x-2) + 0 \times (y-1) + 12 \times \frac{(x-2)^2}{2} + (-6) \times \frac{(y-1)^2}{2} + 0 \times (x-2)(y-1) + \ldots$$
$$= 4 + 9(x-2) + 6(x-2)^2 - 3(y-1)^2 + \ldots$$

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Example 2.4. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x^2u_{xx} + 2xyu_{xy} + y^2 = (1 - 4\sin^2 u)\sin 2u$

Sol. Let, $v = \frac{x^3+y^3}{x-y} = \tan u$. Here, v is a homogeneous function of x and y of degree 2. Then by Euler's theorem,

$$\begin{aligned} & x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v = 2\tan u \\ \Rightarrow\ & x\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = 2v = 2\tan u \\ \Rightarrow\ & \Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big)\sec^2 u = 2\tan u \\ \Rightarrow\ & x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \end{aligned}$$

Differentiating with respect to x and y respectively,

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x} \tag{2.1}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\frac{\partial u}{\partial y} \tag{2.2}$$

Now $2.1 \times x + (2.2) \times y$ we get,

$$\begin{aligned} x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + \sin 2u &= 2\cos 2u\sin 2u \\ &= \sin 2u(2\cos 2u - 1) \\ &= (1 - 4\sin^2 u)\sin 2u \end{aligned}$$

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3 Chapter 2: Integral Calculus

$$I = \int_a^b f(x)\,dx$$

If the integrand of the integral becomes infinite in the interval $[a, b]$ or f has points of infinite discontinuity or when both or either one of the limit points/ end points $[a, b]$, is infinite then the integral I is called improper integral.

Example 3.1. $\int_0^1 \frac{1}{x^2}dx, \int_1^2 \frac{1}{(2-x)(1-x)}dx, \int_0^1 \frac{1}{(x-\frac{1}{2})^2}dx, \int_0^\infty e^x dx, \int_{-\infty}^0 \frac{1}{e^x}dx$

Convergence at the left end point.

$$I = \int_a^b f(x)dx$$

Let, at $x = a$, $f(x)$ has infinite discontinuity. In this case, we first find the value of $\int_{a+\varepsilon}^b f(x)\,dx$ then pass the limit $\varepsilon \to 0^+$ i.e,

$$I = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f(x)\,dx$$

Convergence at the right end point.

$$I = \int_a^b f(x)dx$$

Let, at $x = b$, $f(x)$ has infinite discontinuity. In this case, we first find the value of $\int_a^{b+\varepsilon} f(x)\,dx$ then pass the limit $\varepsilon \to 0^-$ i.e,

$$\begin{aligned} I &= \lim_{\varepsilon\to 0^-} \int_a^{b+\varepsilon} f(x)dx \\ &= \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f(x)dx \end{aligned}$$

Convergence inside the interval. Let at $x = c \in (a, b), f(x)$ has an infinite discontinuity. In this case, we first separate $I = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, then we do the same as before

$$\begin{aligned} I &= \int_a^b f(x)\,dx \\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{\varepsilon\to 0^-} \int_a^{c+\varepsilon} f(x)\,dx + \lim_{\varepsilon\to 0^+} \int_{c+\varepsilon}^b f(x)\,dx \end{aligned}$$

Convergence at $+\infty, -\infty$.

$$\begin{aligned} I &= \int_a^\infty f(x)\,dx \\ &= \lim_{t\to\infty} \int_a^t f(x)\,dx \\ I_1 &= \int_{-\infty}^b f(x)\,dx \\ &= \lim_{t\to-\infty} \int_t^b f(x)\,dx \\ I_2 &= \int_{-\infty}^\infty f(x)\,dx \\ &= \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx \end{aligned}$$

Example 3.2. $I = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Sol. At $x = 1$, there is an infinite discontinuity.

$$\begin{aligned} I &= \lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx \\ &= \lim_{\varepsilon\to 0^+} \left[\sin^{-1} x\right]_0^{1-\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \sin^{-1}(1-\varepsilon) \\ &= \sin^{-1} 1 \\ &= \frac{\pi}{2} \end{aligned}$$

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Example 3.3. $I = \displaystyle\int_0^{\pi} \frac{1}{\sin x}\, dx$

Hint.

$$\begin{aligned} I &= \int_0^{\pi} \frac{1}{\sin x}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sin x}\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sin x}\, dx \end{aligned}$$

Example 3.4. $I = \displaystyle\int_0^{\infty} \frac{1}{1+x^2}\, dx$

Sol.

$$\begin{aligned} I = \int_0^{\infty} \frac{1}{1+x^2}\, dx = \lim_{t\to\infty} \int_0^{t} \frac{1}{1+x^2}\, dx &= \lim_{t\to\infty} \Big[\tan^{-1} x\Big]_0^t \\ &= \lim_{t\to\infty} \tan^{-1} t \\ &= \frac{\pi}{2} \end{aligned}$$

Example 3.5. $I = \displaystyle\int_a^b \frac{1}{(x-a)^n}\, dx$

Sol.

$$\begin{aligned} I &= \int_a^b \frac{1}{(x-a)^n}\, dx \\ &= \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} \frac{1}{(x-a)^n}\, dx \\ &= \lim_{\varepsilon\to 0^+} \Big[\frac{1}{1-n}(x-a)^{(1-n)}\Big]_{a+\varepsilon}^{b} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{1-n}\Big(\frac{1}{(b-a)^{(n-1)}} - \frac{1}{\varepsilon^{n-1}}\Big) \end{aligned}$$

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It is convergent if $n < 1$ and divergent if $n \geq 1$.